Note on Totally Skew Embeddings of Quasitoric Manifolds over Cube

Đorđe Baralić
Mathematical Institute SASA
Belgrade, Serbia

Abstract

Skew embeddings are introduced by Ghomi and Tabachnikov in [7]. They are naturally related to classical problems in topology, such as the generalized vector field problem and the immersion problem for real projective spaces. In recent paper [1], totaly skew embeddings are studied by using the topological obstruction theory. In the same paper it is conjectured that for every n-dimensional, compact smooth manifold M^n $(n > 1)$,

$$N(M^n) \leq 4n - 2\alpha(n) + 1,$$

where $N(M^n)$ is defined as the smallest dimension N such that there exists a *totally skew* embedding of a smooth manifold M^n in $\mathbb{R}^N$.

We prove that for every n, there is a quasitoric manifold Q^{2n} for which the orbit space of T^n action is a cube I^n and

$$N(Q^{2n}) \geq 8n - 4\alpha(n) + 1.$$

Using the combinatorial properties of cohomology ring $H^*(Q^{2n}, \mathbb{Z}_2)$, we construct an interesting general non-trivial example different from known example of the product of complex projective spaces.

1 Introduction

The studying of skew embeddings was started by Ghomi and Tabachnikov in [7]. Recall, that two lines in an affine space $\mathbb{R}^N$ are called *skew* if they are neither parallel nor have a point in common or equivalently if their affine span has dimension 3. More generally, affine subspaces $U_1, \ldots, U_l$ of $\mathbb{R}^N$ are called *skew* if their affine span has dimension $\dim(U_1) + \cdots + \dim(U_l) + l - 1$, in particular a pair U, V of affine subspaces of $\mathbb{R}^N$ is skew if and only if each two lines $p \subset U$ and $q \subset V$ are skew. An embedding $f : M^n \to \mathbb{R}^N$ of a smooth manifold is called *totally skew* if for each two distinct points $x, y \in M^n$ the affine subspaces $df(T_xM)$ and $df(T_yM)$ of $\mathbb{R}^N$ are skew. Define $N(M^n)$ as the smallest N such that there exists a totally skew embedding of M^n into $\mathbb{R}^N$.

Ghomi and Tabachnikov established a surprising connection of $N(M^n)$ with some classical invariants of geometry and topology. For example they showed [7, Theorem 1.4] that the problem of estimating $N(\mathbb{R}^n)$ is intimately related to the generalized vector field problem and the immersion problem for real projective spaces, as

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exemplified by the inequality

$$N(\mathbb{R}^n) \geq r(n) + n$$

where $r(n)$ is the minimum r such that the Whitney sum $r\xi_{n-1}$ of r copies of the canonical line bundle over $\mathbb{R}P^{n-1}$ admits $n+1$ linearly independent continuous cross-sections.

Another example ([7, Theorem 1.2]) is the inequality

$$N(S^n) \leq n + m(n) + 1$$

where $m(n)$ is an equally well-known function defined as the smallest m such that there exists a non-singular, symmetric bilinear form $B : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^m$. As a consequence they deduced the inequalities $N(S^n) \leq 3n + 2$ and $N(S^{2k+1}) \leq 3(2k + 1) + 1$.

Up to our knowledge, the exact values of $N(M^n)$ are known only for $N(\mathbb{R}^1) = 3$, $N(S^1) = 4$ and $N(\mathbb{R}^2) = 6$. General upper and lower bounds are given by Ghomi and Tabachnikov inequality

$$2n + 2 \leq N(M^n) \leq 4n + 1. \tag{1}$$

In the papers [10], [11] and [12] some more general conditions with multiple regularity are studied.

In the paper [1], slightly different approach to the invariant $N(M^n)$ is used. Using the topological obstruction theory the lover bound is improved for various classes of manifolds, such as projective spaces (both real and complex), products of projective spaces, Grassmannians, etc. Stiefel-Whitney classes are obstructions to totaly skew embeddings and it is shown [1, Proposition 1.] and [1, Corollary 4.]

Theorem 1.1. *If* $k := \max\{i \mid \overline{w}_i(M) \neq 0\}$ *then*

$$N(M) \geq 2n + 2k + 1.$$

In the same paper, some evidence in favor the conjecture [1, Conjecture 20.]

$$N(M^n) \leq 4n - 2\alpha(n) + 1,$$

for compact smooth manifold M^n $(n > 1)$, where $\alpha(n)$ is the number of non-zero digits in the binary representation of n. R. Cohen [4] in 1985 resolved positively the famous *Immersion Conjecture*, predicted that any compact smooth n-manifold for $n > 1$ can be immersed in $\mathbb{R}^{2n-\alpha(n)}$.

Quasitoric manifolds are a class of manifolds with well understood cohomology ring which is determined by Davis-Januszkiewicz formula [6, Theorem 4.14, Corollary 6.8]. Other topological invariants could be computed from the formula, and we are particulary interested in Stiefel-Whitney clases. In monography [3] by Buchstaber and Panov there is a nice exposition on quasitoric manifolds and its combinatorial and geometrical properties.

Let P be a simple polytope of dimension n with m facets and M a quasitoric manifold of dimension $2n$ over P. Let v_j ($\deg v_j = 2, j = 1, \ldots, m$) be Poincaré dual of codimension two invariant submanifold M_j in M^{2n}, thus to each facet F_j we assign v_j because the image of the characteristic submanifold M_j of the orbit projection $M \to P$

is the facet F_j. The equivariant cohomology ring $H^*_{T^n}(M;\mathbb{Z}) = H^*(ET \times_{T^n} M)$ of M has the following ring structure:

$$H^*_{T^n}(M;\mathbb{Z}) \simeq \mathbb{Z}[v_1,\ldots,v_m]/\mathcal{I},$$

where $\mathcal{I}$ is the Stanley-Reisner (the face) ideal of polytope P in the polynomial ring $\mathbb{Z}[v_1,\ldots,v_m]$.

Let $\pi : ET \times_T M \to BT$ be the natural projection. The induced homomorphism

$$\pi^* : H^*(BT) = \mathbb{Z}[t_1,\ldots,t_n] \to H^*(ET \times_{T^n} M) = H^*_{T^n}(M;\mathbb{Z})$$

could be described by a $n \times m$ matrix $\Lambda = (\lambda_1,\ldots,\lambda_m)$, where $\lambda_j \in \mathbb{Z}^n$ ($j = 1,\ldots,m$) corresponds to the generator of Lie algebra isotropy subgroup of characteristic submanifold M_j. The matrix Λ is called *characteristic matrix* of M. Put $\lambda_j = (\lambda_{1j},\ldots,\lambda_{nj})^t \in \mathbb{Z}^n$. Then we have

$$\pi^*(t_i) = \sum_{j=1}^{m} \lambda_{ij} v_j$$

and let $\mathcal{J}$ be the ideal in $\mathbb{Z}[v_1,\ldots,v_m]$ generated by $\pi^*(t_i)$ for all $i = 1,\ldots,n$. The ordinary cohomology of quasitoric manifolds has the following ring structure:

$$H^*(M) \simeq \mathbb{Z}[v_1,\ldots,v_m]/(\mathcal{I}+\mathcal{J}).$$

The Stiefel-Whitney class can be described by the following *Davis-Januszkiewicz formula*:

$$\omega(M) = \imath^* \prod_{i=1}^{m}(1+v_i),$$

where $\imath$ is the inclusion $\imath : M \to ET \times_T M$ and $\imath^*$ is the induced homomorphism.

In Section 2 we describe one special quasitoric manifold M_I over cube I^n by matrix Λ_{M_I}. We describe its cohomology ring and calculate the Stiefel-Whitney class.

Section 3 is devoted to calculation of the Stiefel-Whitney class of normal bundle using the smart manipulations of binomial coefficients in cohomology ring (with $\mathbb{Z}_2$ coefficients). We calculate the obstruction to totally skew embedding of manifold M_I and get the main result of the paper.

2 Quasitoric manifold over cube

2.1 Matrix Λ_{M_I} and cube I^n

A quasitoric manifold M is described by two key objects: its orbit polytope P and the characteristic matrix Λ. Two quasitoric manifolds over the same polytope, but with distinct characteristic matrices are different, with non-isomorphic cohomology rings. Although, polytope P with its combinatorics gives a lot of informations on manifold itself, the characteristic matrix Λ is necessary to understand important topological invariants of the quasitoric manifold.

Let Λ be the integer matrix $(n \times m)$ matrix whose i-th column is formed by the coordinates of the facet vector λ_i, $i = 1,\ldots,m$. Every vertex $v \in P^n$ is an intersection

of n facets: $v = F_{i_1} \cap \cdots \cap F_{i_n}$. Let $\Lambda_{(v)} := \Lambda_{(i_1,\ldots,i_n)}$ be the maximal minor of Λ formed by the columns $i_1, \ldots, i_n$. Then

$$|\det \Lambda| = 1.$$

In other words, to every facet F_i there is an assigned vector $\lambda_i = (\lambda_{1i}, \ldots, \lambda_{ni})^t \in \mathbb{Z}^n$ in such way that for every vertex $v = F_{i_1} \cap \cdots \cap F_{i_n}$ vectors $\lambda_{i_1}, \ldots, \lambda_{i_1}$ are basis of lattice $\mathbb{Z}^n$ (see Figure 1)

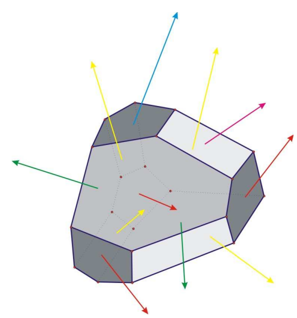

Figure 1: Quasitoric manifold over polytope

In case when P is a rational polytope and $\lambda_i \perp F_i$, for every $i = 1, \ldots, m$ manifold M is a toric variety (see Figure 2).

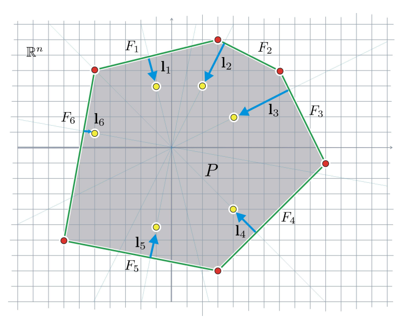

Figure 2: Toric variety

In monograph [3, Construction 5.12] is described a construction of quasitoric manifold from characteristic pair (P^n, l) that is from a combinatorial polytope P and matrix Λ.

Let I^n be a cube and M_{I^n} a quasitoric manifold over I^n. Cube has $2n$ facets F_1, $\ldots$, F_n, F'_1, $\ldots$, F'_n such that $F_i \cap F'_i = \emptyset$ for every $i = 1, \ldots, n$. Let v_1, $\ldots$, v_n, u_1, $\ldots$, u_n be Poincaré duals to characteristic submanifolds over the facets F_1, $\ldots$, F_n, F'_1, $\ldots$, F'_n respectively. Then Stanley-Reisner ideal is generated by

$$\mathcal{I} = \{v_1u_1, v_2u_2, \ldots, v_nu_n\}.$$

We study special quasitoric manifold M_{I^n} over the cube, such that vector λ_j assigned to the facet F_j (or the generators of Lie algebra isotropy subgroup of characteristic submanifold M_j) is $\lambda_j = (\underbrace{0, \ldots, 0}_{i-1}, 1, \underbrace{0, \ldots, 0}_{n-i})^t$ for every $j = 1, \ldots, n$ and vector

λ_{j+n} assigned to the facet F'_j is $\lambda_{n+j} = (\underbrace{1,\dots,1}_{i},\underbrace{0,\dots,0}_{n-i})^t$ for every $j = 1,\dots,n$.

Then we have:

$$\Lambda_{M_I} = \begin{bmatrix} 0 & 0 & \dots & 1 & 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \dots & 0 & 1 & 1 & \dots & 0 \\ 1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{bmatrix}.$$

Ideal $\mathcal{J}$ in $\mathbb{Z}[v_1,\dots,v_n,u_1,\dots,u_n]$ is generated by linear forms

$$\begin{array}{l} v_1 + u_1, \\ v_2 + u_1 + u_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots , \\ v_n + u_1 + u_2 + \cdots + u_n. \end{array}$$

2.2 Cohomology ring $H^*(M_I)$ and Stiefel-Whitney class $\omega(M_I)$

Cohomology ring $H^*(M_I)$ is determined using Davis-Januszkiewicz theorem:

Proposition 2.1. *Cohomology ring $H^*(M_I;\mathbb{Z})$ is isomorphic to*

$$H^*(M_I;\mathbb{Z}) \simeq \mathbb{Z}[u_1,\dots,u_n]/\mathcal{F}$$

where $\mathcal{F}$ is ideal in polynomial ring $\mathbb{Z}[u_1,\dots,u_n]$ (such that $\mathrm{deg} u_1 = \cdots = \mathrm{deg} u_n = 2$*) generated by quadratic forms*

$$\mathcal{F} = \{u_1^2, u_2^2 + u_1u_2, \dots, u_n^2 + u_1u_n + u_2u_n + \cdots + u_{n-1}u_n\}.$$

It is easy to show the following relations in $H^*(M_I;\mathbb{Z})$:

Proposition 2.2. *For every $i = 1,\dots,n$ holds*

$$u_i^i \neq 0 \quad \textit{and } u_i^{i+1} = 0.$$

Proposition 2.3. *For every $i = 2,\dots,n$ holds*

$$(1+u_i)(1+v_i) = 1 + u_1 + \cdots + u_{i-1}$$

By Universal Coefficient Theorem we obtain that

$$H^*(M_I;\mathbb{Z}_2) \simeq \mathbb{Z}_2[u_1,\dots,u_n]/\mathcal{F}$$

where $\mathcal{F}$ is ideal in polynomial ring $\mathbb{Z}_2[u_1,\dots,u_n]$ (such that $\mathrm{deg} u_1 = \cdots = \mathrm{deg} u_n = 2$) generated by quadratic forms

$$\mathcal{F} = \{u_1^2, u_2^2 + u_1u_2, \dots, u_n^2 + u_1u_n + u_2u_n + \cdots + u_{n-1}u_n\}.$$

Stiefel-Whitney class is the characteristic class in cohomology with $\mathbb{Z}_2$ coefficients. By Davis-Januszkiewicz formula Stiefel-Whitney class of M_I is given by

$$\omega(M_I) = (1+u_1)\cdots(1+u_n)(1+v_1)\cdots(1+v_n),$$

but by using Propositions 2.1 and 2.3 it is easily reduced to

$$\omega(M_I) = (1+u_1)(1+u_1+u_2)\cdots(1+u_1+\cdots+u_{n-1}).$$

For the purposes of the main theorem, we are going to use another form of cohomology ring $H^*(M_I;\mathbb{Z}_2)$. If we choose another generators $t_1, \dots, t_n$ such that

$$\begin{array}{lllllllll} t_1 & = & u_1, & & & & & & \\ t_2 & = & u_1 & + & u_2, & & & & \\ \multicolumn{8}{l}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\quad ,} & \\ t_n & + & u_1 & + & u_2 & + & \cdots & + & u_n, \end{array}$$

we get that

$$H^*(M_I;\mathbb{Z}_2) \simeq \mathbb{Z}_2[t_1,\dots,t_n]/\mathcal{G}$$

where $\mathcal{G}$ is ideal in polynomial ring $\mathbb{Z}_2[t_1,\dots,t_n]$ (such that $\deg t_1 = \cdots = \deg t_n = 2$) generated by quadratic forms

$$\mathcal{G} = \{t_1^2, t_2^2 + t_1t_2, \dots, t_n^2 + t_{n-1}t_n\}.$$

Consequently, total Stiefel-Whitney class is given by

$$\omega(M_I) = (1+t_1)\cdots(1+t_{n-1}).$$

It is nod hard to check that the following is true in $H^*(M_I;\mathbb{Z}_2)$:

Proposition 2.4. *For every $i = 1,\dots,n$ holds*

$$t_i^i = t_1t_2\cdots t_i \neq 0 \quad \textit{and } t_i^{i+1} = 0.$$

3 Topological obstructions to totally skew embeddings of manifold M_I

3.1 Stiefel-Whitney class $\overline{\omega}(M_I)$ of normal bundle

For the purposes of the main theorem, we are interested in characteristic classes $\overline{\omega}(M_I)$ of the normal bundle. Stiefel-Whitney classes $\omega(M_I)$ and $\overline{\omega}(M_I)$ are related to each other by equality

$$\omega(M_I)\cdot\overline{\omega}(M_I) = 1.$$

In the previous section the Stiefel-Whitney classes $\omega(M_I)$ is determined. So, by Proposition 2.4, it holds:

Lemma 3.1. *Total Stiefel-Whitney class $\overline{\omega}(M_I)$ of the normal bundle is given by:*

$$\overline{\omega}(M_I) = (1+t_1)(1+t_2+t_2^2)\cdots(1+t_{n-1}+\cdots+t_{n-1}^{n-1}).$$

Since $\overline{\omega}_{2i}(M_I) = 0$ when $i \geq n$, it is far from obvious what is $\overline{\omega}(M_I)$ in cohomology ring $H^*(M_I;\mathbb{Z}_2)$. For small n, we could calculate $\overline{\omega}(M_I)$ by hand:

Exercise 1.

1. $\overline{\omega}(M_{I^2}) = 1+t_1$,
2. $\overline{\omega}(M_{I^3}) = 1+(t_1+t_2)$,

3. $\overline{\omega}(M_{I^4}) = 1 + (t_1 + t_2 + t_3) + t_1t_3 + t_1t_2t_3$,

4. $\overline{\omega}(M_{I^5}) = 1 + (t_1 + t_2 + t_3 + t_4) + (t_1t_3 + t_1t_4 + t_2t_4) + (t_1t_2t_3 + t_2t_3t_4)$.

By Lemma 3.1 for total Stiefel-Whitney classes of $\overline{\omega}(M_{I^n})$ and $\overline{\omega}(M_{I^{n+1}})$ the following recurrence relation holds (in $H^*(M_{I^{n+1}};\mathbb{Z}_2)$):

$$\overline{\omega}(M_{I^{n+1}}) = \overline{\omega}(M_{I^n})(1 + t_n + \cdots + t_n^n), \tag{2}$$

or more explicitly

$$\overline{\omega}_{2k}(M_{I^{n+1}}) = \overline{\omega}_{2k}(M_{I^n}) + t_n\,\overline{\omega}_{2k-2}(M_{I^n}) + \cdots + t_n^k \text{ for all } k = 0, \ldots, n-1 \tag{3}$$

and

$$\overline{\omega}_{2n}(M_{I^{n+1}}) = t_n\,\overline{\omega}_{2n-2}(M_{I^n}) + \cdots + t_n^n. \tag{4}$$

Define numbers σ_n^k for all positive integers n and $0 \le k \le n-1$ as follows

$$\sigma_n^k = \begin{cases} 1 & \text{if the total number of the distinct monomials in } \overline{\omega}_{2k}(M_{I^n}) \text{ is odd} \\ 0 & \text{elsewhere} \end{cases} \tag{5}$$

So by 3 and 4, it holds

$$\sigma_{n+1}^k = \sum_{i=0}^{k} \sigma_n^i \tag{6}$$

for every $k = 1, \ldots, n-1$ and

$$\sigma_{n+1}^n = \sigma_{n+1}^{n-1}.$$

Let us write the first n rows of numbers σ_n^k for $k = 0, \ldots, n$:

$$\begin{array}{cccccccc}
1 \\
1 & 1 \\
1 & 0 & 0 \\
1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\multicolumn{8}{c}{\cdots\cdots\cdots\cdots\cdots\cdots}
\end{array}$$

The previous sequence is closely related to the following sequence of binomial coefficients $\binom{n+k}{k}$:

$$\begin{array}{cccccccc}
\boxed{1} \\
\boxed{1} & \boxed{3} \\
\boxed{1} & 4 & 10 \\
\boxed{1} & \boxed{5} & \boxed{15} & \boxed{35} \\
\boxed{1} & 6 & \boxed{21} & 56 & 70 \\
\boxed{1} & \boxed{7} & \binom{8}{2} & \binom{9}{3} & \binom{10}{4} & \binom{11}{5} \\
\boxed{1} & 8 & \binom{9}{2} & \binom{10}{3} & \binom{11}{4} & \binom{12}{5} & \binom{13}{6} \\
\boxed{1} & \boxed{9} & \boxed{\binom{10}{2}} & \boxed{\binom{11}{2}} & \boxed{\binom{12}{4}} & \boxed{\binom{13}{5}} & \boxed{\binom{14}{6}} & \boxed{\binom{15}{7}} \\
\multicolumn{8}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots}
\end{array}$$

Easy mathematical induction shows that:

Lemma 3.2.

$$\sigma_n^k \equiv \binom{n+k}{k} \pmod 2.$$

By previous Lemma, in the case when $n = 2^r$ we have

$$\sigma_n^{n-1} \equiv \binom{2^r + (2^r - 1)}{2^r - 1} \equiv \binom{2^{r+1} - 1}{2^r - 1} \equiv 1 \pmod 2.$$

Obviously, by the definition of σ_n^k if $\sigma_n^k = 1$ then

$$\overline{\omega}_{2k}(M_{I^n}) \neq 0.$$

Thus, we have:

Theorem 3.1. *If $n = 2^r$ is a power of two then*

$$\overline{\omega}_{2n-2}(M_{I^n}) = t_1 t_2 \cdots t_{n-1} \neq 0.$$

Corollary 3.1. *For $n = 2^r$, quasitoric manifold M_{I^n} cannot be totally skew embedded in $\mathbb{R}^N$ when N is less than*

$$8n - 3 = 4 \cdot \dim M_{I^n} - 3.$$

3.2 Topological obstructions when n is not a power of 2

Theorem 3.1 is the sharpest possible result that one could obtain using Stiefel-Whitney classes for quasitoric manifolds. However, when n is not a power of 2 the previously constructed quasitoric manifold M_{I^n} in general does not achieve the maximal possible value k for which Stiefel-Whitney class $\overline{\omega}_{2k}(M_{I^n}) \neq 0$.

This problem could be overcome using the results from the previous part.

Let $n = 2^{r_1} + 2^{r^2} + \cdots + 2^{r_t}$, $r_1 > r_2 > \cdots > r_t \geq 0$ be the binary representation of n and let $m_i = 2^{r_i}$ for $i = 1, \ldots, t$. Divide the facets of a cube I_n into t groups $A_1, \cdots, A_t$ in such way that the opposite facets belong to the same group and $|A_j| = 2m_j$ for $j = 1, \ldots, t$. For every $j = 1, \ldots, t$, denote the facets from A_j with $F_i^{(j)}$ and $F_i'^{(j)}$ (the opposite facets), $i = 1, \ldots, m_j$. We are going to construct new quasitoric manifold Q^{2n} over cube by defining a new characteristic matrix Λ. Let $\lambda_i^{(j)} = (\underbrace{0, \ldots, 0,}_{(\sum_s^{j-1} m_s)+(i-1)} \; 1, \; \underbrace{0, \ldots, 0}_{n-(\sum_s^{j-1} m_s)-i})^t \in \mathbb{Z}^n$ and $\lambda_{i+n}^{(j)} = (\underbrace{0, \ldots, 0}_{(\sum_s^{j-1} m_s)} \underbrace{1, \ldots, 1,}_{i} \underbrace{0, \ldots, 0}_{n-(\sum_s^{j-1} m_s)-i})^t \in \mathbb{Z}^n$ be the vectors assigned to the facets $F_i^{(j)}$ and $F_i'^{(j)}$ respectively. Let $v_i^{(j)}$ and $u_i^{(j)}$ be Poincaré duals to the characteristic submanifolds over the facets $F_i^{(j)}$ and $F_i'^{(j)}$ respectively, for all facets. Then the characteristic matrix Λ is:

$$\Lambda = \left[\begin{array}{c|ccc} & \begin{bmatrix} 0 \end{bmatrix} & \cdots & \begin{bmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{bmatrix}_{m_t \times m_t} \\ & \begin{bmatrix} 0 \end{bmatrix} & \cdots & \begin{bmatrix} 0 \end{bmatrix} \\ I_n & \vdots & \ddots & \vdots \\ & \begin{bmatrix} 0 \end{bmatrix} & \cdots & \begin{bmatrix} 0 \end{bmatrix} \\ & \begin{bmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{bmatrix}_{m_1 \times m_1} & \cdots & \begin{bmatrix} 0 \end{bmatrix} \end{array}\right].$$

So by Davis-Januszkiewicz theorem we get:

Theorem 3.2.

- *Cohomology ring $H^*(Q;\mathbb{Z})$ is isomorphic to*

$$H^*(Q;\mathbb{Z}) \simeq \mathbb{Z}[u_1^{(1)}, \dots, u_{m_1}^{(1)}, \dots, u_{m_t}^{(t)}]/\mathcal{F}$$

where $\mathcal{F}$ is ideal in polynomial ring $\mathbb{Z}[u_1^{(1)}, \dots, u_{m_1}^{(1)}, \dots, u_{m_t}^{(t)}]$ (such that $\deg u_i^{(j)} = 2$ for all j and i) generated by quadratic forms

$$\mathcal{F} = \{{u_1^{(j)}}^2, {u_2^{(j)}}^2 + u_1^{(j)} u_2^{(j)}, \dots, {u_{m_j}^{(j)}}^2 + u_1^{(j)} u_{m_j}^{(j)} + u_2^{(j)} u_{m_j}^{(j)} + \dots + u_{m_j-1}^{(j)} u_{m_j}^{(j)} | j \in [t]\}.$$

- *Cohomology ring $H^*(Q;\mathbb{Z}_2)$ is isomorphic to*

$$H^*(Q;\mathbb{Z}_2) \simeq \mathbb{Z}_2[u_1^{(1)}, \dots, u_{m_1}^{(1)}, \dots, u_{m_t}^{(t)}]/\mathcal{G}$$

where $\mathcal{G}$ is ideal in polynomial ring $\mathbb{Z}_2[u_1^{(1)}, \dots, u_{m_1}^{(1)}, \dots, u_{m_t}^{(t)}]$ (such that $\deg u_i^{(j)} = 2$ for all j and i) generated by quadratic forms

$$\mathcal{G} = \{{u_1^{(j)}}^2, {u_2^{(j)}}^2 + u_1^{(j)} u_2^{(j)}, \dots, {u_{m_j}^{(j)}}^2 + u_1^{(j)} u_{m_j}^{(j)} + u_2^{(j)} u_{m_j}^{j} + \dots + u_{m_j-1}^{(j)} u_{m_j}^{(j)} | j \in [t]\}.$$

- *Total Stiefel-Whitney class $\omega(Q)$ is given by*

$$\omega(Q) = \prod_{j=1}^{t} (1 + u_1^{(j)})(1 + u_1^{(j)} + u_2^{(j)}) \cdots (1 + u_1^{(j)} + \dots + u_{m_j-1}^{(j)}).$$

In the same fashion as in the previous section we choose new generators $t_1^{(1)}, \dots, t_{m_1}^{(1)}, \dots, t_{m_t}^{(t)}$ such that

$$H^*(Q;\mathbb{Z}_2) \simeq \mathbb{Z}_2[t_1^{(1)}, \dots, t_{m_1}^{(1)}, \dots, t_{m_t}^{(t)}]/\mathcal{G}$$

where $\mathcal{G}$ is ideal in polynomial ring $\mathbb{Z}_2[t_1^{(1)}, \dots, t_{m_1}^{(1)}, \dots, t_{m_t}^{(t)}]$ (such that $\deg t_1 = \dots = \deg t_n = 2$) generated by quadratic forms

$$\mathcal{G} = \{{t_1^{(j)}}^2, {t_2^{(j)}}^2 + t_1^{(j)} t_2^{(j)}, \dots, {t_{m_j}^{(j)}}^2 + t_{m_j-1}^{(j)} t_{m_j}^{(j)} | j \in [t]\}.$$

Consequently, total Stiefel-Whitney class is given by

$$\omega(Q) = \prod_{j=1}^{t}(1+t_1^{(j)})\cdots(1+t_{m_j-1}^{(j)}).$$

Thus, the corresponding dual Stiefel-Whitney class is given by

$$\overline{\omega}(Q) = \prod_{j=1}^{t}(1+t_1^{(j)})(1+t_2^{(j)}+{t_2^{(j)}}^2)\cdots(1+t_{m_j-1}^{(j)}+\cdots+{t_{m_j-1}^{(j)}}^{m_j-1}).$$

But, according Theorem 3.1 we have:

$$\overline{\omega}(Q) = \prod_{j=1}^{t}(1+(t_1^{(j)}+\cdots+t_{m_j-1}^{(j)})+\cdots+t_1^{(j)}t_2^{(j)}\cdots t_{m_j-1}^{(j)}).$$

So we proved that the highest nontrivial dual Stiefel-Whitney class is

$$\overline{\omega}_{2n-2\alpha(n)}(Q) = t_1^{(1)}\cdots t_{m_1-1}^{(1)}t_1^{(2)}\cdots t_{m_t-1}^{(t)},$$

where $\alpha(n)$ is the number of non-zero digits in the binary representation of n

As corollary we obtain:

Theorem 3.3 (Main theorem)**.** *For every positive integer n there is a quasitoric manifold over cube I^n such that*

$$N(Q) \geq 8n - 4\alpha(n) + 1.$$

Remark. Similar result cannot be obtained in the class of toric varieties from a cube because Stiefel-Whitney class is trivial in that case.

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ĐORĐE BARALIĆ, Mathematical Institute SASA, Kneza Mihaila 36, p.p. 367, 11001 Belgrade, Serbia

E-mail address: djbaralic@mi.sanu.ac.rs